NONLINEAR OPTICS (PHYC/ECE 568)
Spring 2020 - Instructor: M. Sheik-Bahae
University of New Mexico
Homework #4, Due Monday, March 23

Problem 1. SHG in KDP:

a. Calculate the type-I phase matching angle for SHG in KDP using 1.06 \mu m output of a Nd:YAG laser.

b. For a beam radius w0=500 \mu m, calculate the aperture length defined as l_a = \sqrt{\pi w_0/\rho} where \rho is the Poynting vector walk-off angle. Obtain the aperture length for w0=15 \mu m and discuss the role of additional limitations that may be imposed due to diffraction of the beam.

Problem 2. SHG Bandwidth:

a. Calculate the bandwidth \Delta \omega associated with a phase-matched SHG process in terms of the group velocities v_g(\omega_1) and v_g(2\omega_1). In the low-depletion approximation, this corresponds to the width of the Sinc^2 function which is taken to be \delta(\Delta kL)=2\pi with L denoting the length of the nonlinear crystal.

Hint: Use the first-order term in the Taylor series expansion of \Delta k(\omega).

b. Discuss how your results in (a) explains the limitation on the SHG-efficiency when ultrashort laser pulses are used.

Problem 3. What about the fundamental wave?

Consider the case of a phase-matchable SHG process; but instead of being concerned about the second-harmonic beam (at 2\omega), we would like to determine the fate of the transmitted fundamental field at \omega (see also problem 2.20 in Boyd, 3rd ed.).

(a) Start with the coupled amplitude equations (i.e. Eqns. 2.7.10-11 in Boyd). Eliminate A_2 to obtain the following nonlinear differential equation for A_1:

\frac{d^2A_1}{dz^2} + i\Delta k \frac{dA_1}{dz} + \Gamma^2 A_1 \left[2 \left|A_1(0)\right|^2 - 1\right] = 0

where \Gamma^2=4d^2\omega_1^2|A_1(0)|^2/(c^2n_1n_2).

(b) Now make the low-depletion approximation by setting |A_1|^2=|A_1(0)|^2 in the above equation. Solve for A_1 for a propagation length L. (Hint: You need a second initial condition that is obtained from E_2(0)=0).

(c) Taking A_1=|A_1|e^{i\phi}, plot |A_1|/|A_1(0)| and \phi versus \Delta kL (from -4\pi to 4\pi) for \Gamma^2L^2=0.1, 0.2, and 0.4. Discuss your results (i.e. sign reversal vs. \Delta k, etc.)

The above process (i.e. the intensity-dependent phase variation of the fundamental wave) has been termed \chi^{(2)}:\chi^{(2)} cascading nonlinearity. It mimics a third order \chi^{(3)} process where \chi^{(3)}(\omega_1\omega_2\omega_3)\chi^{(2)}(2\omega_1\omega_2)\chi^{(2)}(2\omega_2\omega_2\omega_3) is effectively a cascade of two second order effects. The cascading nonlinearity has generated some interest for applications requiring large \chi^{(3)} effects (i.e. optical switching, spatial solitons, and, in general, processes requiring an n_2-type nonlinearity). See Sheik-Bahae and Hasselbeck (OSA Handbook, Chapter 17).
Problem 4. Cascading for THG (Third-Harmonic Generation) in KDP.
Actually, cascading 2nd order effects to obtain an effective third-order effect is not a new concept. In fact
the most efficient way to generate the third-harmonic (3\(\omega\)) of a laser beam is to first produce 2\(\omega\) (in an SHG
process) and then use SFG to generate 3\(\omega\)=2\(\omega\)+\(\omega\). The phase matching requirement, however, dictates that
this cascading be performed in two separate crystals with proper orientation. In Problem 1, you calculated
the phase matching angle (\(\phi_m\)) for type-I SHG in KDP. Now calculate \(\phi_m\) for a second crystal to produce the
third-harmonic of a YAG laser. (Note: No rotation of polarization is used between the two crystals).

\[ n^2 = A + B/(\lambda^2-C) + D\lambda^2/(\lambda^2-E), \lambda \text{ in } \mu\text{m} \]

<table>
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<tr>
<th>Sellmeier Equation</th>
<th>KDP</th>
<th>KD*P</th>
<th>ADP</th>
<th>CDA</th>
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For frequency-doubling (SHG) and tripling (THG) of Nd:YAG laser at 1064 nm, both type I and type II phase-matchings can be employed for KDP and KD*P. In the high power case, the KDP crystals are often used with standard size of 12x12x25 mm³. For frequency-quadrupling (4HG, output at 266 nm) of Nd:YAG laser, KDP crystal is normally recommended.
NLO - PS #3 Solution

problem (1) \rightarrow (8)

SHG in KDP:

for \( \lambda_1 = 1064 \text{ nm} \)
\[
\begin{align*}
&n_0(\lambda_1) = 1.4942 \\
n_e(\lambda_1) = 1.4603
\end{align*}
\]

for \( \lambda_2 = 532 \text{ nm} \)
\[
\begin{align*}
&n_0(\lambda_2) = 1.5129 \\
n_e(\lambda_2) = 1.4709
\end{align*}
\]

for negative uniaxial: \( 0 + 0 \rightarrow e \)

\[
n_0(\lambda_1) = n_e(\lambda_2, \theta_m) = \left[ \frac{\frac{2}{n_e^2(\lambda_2)} \cos \theta_m + \frac{\cos \theta_m}{n_0^2(\lambda_2)}}{\frac{4}{n_e^2(\lambda_2)}} \right]^{1/2}
\]

\[
\frac{1}{n_0^2(\lambda_1)} = \frac{4}{n_e^2(\lambda_2)} \cos \theta_m \left[ \frac{1}{n_e^2(\lambda_2)} - \frac{1}{n_0^2(\lambda_2)} \right] + \frac{1}{n_0^2(\lambda_2)}
\]

\[
\cos \theta_m = \frac{1}{n_0^2(\lambda_1)} - \frac{1}{n_0^2(\lambda_2)}
\]

\[
\cos \theta_m = \left( \frac{1.4942}{1.4709} \right)^2 - \left( \frac{1.4942}{1.5129} \right)^2
\]

\[
\theta_m = 91.3^\circ
\]

\( b \)

\[
\rho = \frac{n_0^2(\lambda_1)}{2} \left[ \frac{1}{n_e^2(\lambda_2)} - \frac{1}{n_0^2(\lambda_2)} \right] \lambda_1 (2 \theta_m)
\]

\[
= \frac{(1.4942)^2}{2} \left[ \left( \frac{1}{1.4709} \right)^2 - \left( \frac{1}{1.5129} \right)^2 \right] \lambda_1 (2 \theta_m)
\]

\[
\rho = 2.8 \text{ mrad}
\]

\[
= 1.6^\circ
\]

\[
\lambda_0 = \frac{\sqrt{2} \omega_0}{\rho}
\]

\[
\text{for } \omega_0 = 500 \text{ mm } : \quad \lambda_0 = 31.6 \text{ mm}
\]

\[
\text{for } \omega_0 = 15 \text{ mm } : \quad \lambda_0 = 0.95 \text{ mm}
\]

For such a small beam waist, we need to consider the size of the crystal!
Problem 1

\[ z_0 = \frac{\pi \omega_0}{\lambda}, \quad l_p = \frac{\sqrt{z_0}}{\lambda} = 0.845 \, \text{cm} \]

radius \[ r = \sqrt{r_0} \, l_a \]

\[ l_{\text{eff}} = \min (l_p, l_a) \]

Problem 2

\[ \Delta K(\omega_1) = \frac{2n (2\omega_1) \omega_1}{c} - \frac{2n (\omega_1) \omega_1}{c} = K(2\omega_1) - 2K(\omega_1) \]

\[ \Delta K(\omega) = \frac{\partial \Delta K}{\partial \omega} \bigg|_{\omega = \omega_1} \]

\[ = \Delta K(\omega_1) + \left[ \frac{\partial K(2\omega)}{\partial \omega} \bigg|_{\omega = 2\omega_1} - 2 \frac{\partial K(\omega)}{\partial \omega} \bigg|_{\omega = \omega_1} \right] (\omega - \omega_1) \]

\[ = \Delta K(\omega_1) + 2 \left[ \frac{1}{V_g(2\omega_1)} - \frac{1}{V_g(\omega_1)} \right] (\omega - \omega_1) \]

(Phase matching for \( \omega_1 \))

Taking \( |\omega - \omega_1| = \frac{\Delta \omega}{2} \)

\[ \delta (\Delta K \mathbf{L}) = \pi \quad \Rightarrow \quad \Delta \omega \mathbf{L} = \frac{\pi}{2} \]

\[ \delta (\Delta K \mathbf{L}) = 2\pi \quad \Rightarrow \quad \Delta \omega \mathbf{L} = \pi \]

\[ \Delta \omega = \frac{\pi}{L} \left[ \frac{1}{V_g(2\omega_1)} - \frac{1}{V_g(\omega_1)} \right]^{-1} \quad \checkmark \]
Problem 2 - (b)

We can also write:

\[ \Delta \omega = \frac{\pi \left( \frac{L}{\beta g(2\omega)} - \frac{L}{\beta g(\omega)} \right)}{\left| 1/t_2 - 1/t_1 \right|} = \frac{\pi \omega}{\beta g(\omega)} \left| 1/t_2 - 1/t_1 \right| \]

- \( t_2 \) and \( t_1 \) are the times that it takes for second harmonic and fundamental to propagate through the crystal separately.
- Because of group-velocity mismatch; \( \beta g(2\omega) \neq \beta g(\omega) \)

Then the fundamental pulse and the generated second harmonic pulse will walk-off from each other if \( \Delta \rho \leq 1/t_2 - 1/t_1 \).
Problem 3

1) \[ \frac{dA_1}{dz} = \frac{\varepsilon_{0} n_1^2 \omega c A_2 A_1^* e^{-i\beta k z}}{k_c \varepsilon_0} = \frac{q}{n_1} A_2 A_1^* e^{-i\beta k z}; \quad q = \frac{i \pi m_n d}{c} \]

2) \[ \frac{dA_2}{dz} = \frac{q}{n_2^2} A_1^2 e^{+i\beta k z}; \quad \beta k = 2\mu - k_2 \]

From (1) \( \Rightarrow \frac{dA_1}{dz} e^{+i\beta k z} = \frac{q}{n_1} A_2 A_1^* \) \( \Rightarrow \) take \( \int \frac{d}{dz} \rightarrow \)

\[ (\frac{d^2 A_1}{dz^2} + i\beta k \frac{dA_1}{dz}) e^{+i\beta k z} = \frac{q}{n_1} \left[ \frac{dA_2}{dz} A_1^* + A_2 \frac{dA_1^*}{dz} \right] \]

Substituting \( \frac{dA_1^*}{dz} = \frac{dA_2}{dz} \) from (1) \& (2):

\[ (\frac{d^2 A_1}{dz^2} + i\beta k \frac{dA_1}{dz}) e^{+i\beta k z} = \frac{q}{n_1} \left[ \frac{q}{n_2} |A_1|^2 A_1^* + \frac{q^*}{n_1} |A_2|^2 A_1^* \right] e^{+i\beta k z} \]
\[
\begin{align*}
A(z) &= \begin{cases} 
A_1(z_1) & \text{if } |z_1| > |A_0| \\
A_2(z_2) & \text{if } |z_2| > |A_0| \\
0 & \text{otherwise}
\end{cases} \\
A_1(0) &= \begin{cases} 
A_0 & \text{if } A_1 \neq 0 \\
0 & \text{otherwise}
\end{cases} \\
A_2(0) &= \begin{cases} 
A_0 & \text{if } A_2 \neq 0 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Initial conditions:
\[
\begin{align*}
A_1(0) &= 1 \\
A_2(0) &= 0
\end{align*}
\]

To eliminate \( A_2 \), we'll impose:
\[
A_2 = \frac{m_2}{m_1} A_1
\]

Using \( E + I_2 = I_1(0) \)}
Now, let: $\varepsilon = L - \delta$

$$\frac{A_1(L)}{A_1(0)} = \left[ \cos\left(\sqrt{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2}\right) + i \frac{\delta k L}{2} \right] \sin\left(\sqrt{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2}\right) e^{i \frac{\delta k L}{2}}$$

where, $\Pi^2 = \eta^2 |A_1(0)|^2$.

(c)

Let $A_1(\varepsilon) = |A_1(\varepsilon)| e^{i \phi_1(\varepsilon)}$ and $\phi_1(0) = 0$

$$\frac{|A_1(\varepsilon)|}{|A_1(0)|} = \left[ \cos^2\left(\sqrt{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2}\right) + \frac{\left(\frac{\delta k L}{2}\right)^2}{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2} \right]^{1/2}$$

$$\lim_{\varepsilon \to 0} \frac{\phi_1(\varepsilon)}{\varepsilon} = \tan^{-1}\left[ \frac{\left(\frac{\delta k L}{2}\right)}{\sqrt{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2}} \tan\left(\sqrt{\left(\frac{\delta k L}{2}\right)^2 + \Pi^2 L^2}\right) \right] - \frac{\delta k L}{2}.$$
Cascading $\chi(2):\chi(2)$ effect

$\beta := -15, -14.93, -15$

$A(\beta, \gamma) := \cos \left( \sqrt{\frac{2 + \beta^2}{4}} \right) + i \frac{\beta}{2} \sin \left( \sqrt{\frac{2 + \beta^2}{4}} \right)$

$\text{abs}(x) := \begin{cases} g & \text{if } x \geq 0 \\ g & \text{if } x < 0 \end{cases}$

$A_m(\beta, \gamma) = \left( A(\beta, \gamma) - A(\beta, \gamma) \right)^{-1}$

$f(\beta, \gamma) := \left( \frac{\text{arg}(A(\beta, \gamma)) - \frac{\pi}{2}}{2} \right)$

$\text{Phase}(\beta, \gamma) := f(\beta, \gamma) + 1 \cdot \text{sign}(\beta) \cdot 2 \cdot \text{floor} \left( \frac{\text{abs}(f(\beta, \gamma))}{1.99} \right)$

subtracted the pi phase jumps associated with arctan operations

Note: in a non-linear system $\beta < 0$ in normal dispersion regime.
Problem 4

Since no polarization rotation is allowed, SFG in the second crystal has to be Type II.

KDP, is a negative uniaxial crystal, then,

\[ \omega_1 n_1^e (\theta) + 2 \omega_1 n_2^0 = 3 \omega_1 n_3^e (\theta) \]

\[
\frac{1}{\sqrt{\frac{\omega_1^2 \partial_m^2}{(n_1^0)^2} + \frac{\omega_1^2 \partial_m^2}{(n_1^e)^2}}} + 2n_2^0 = \frac{3}{\sqrt{\frac{\omega_1^2 \partial_m^2 + \partial_m^2}{(n_3^0)^2} + \frac{\omega_1^2 \partial_m^2}{(n_3^e)^2}}}.
\]

\[
\begin{aligned}
& n_1^0 = 1.4942 \\
& n_1^e = 1.4603 \\
& n_2^0 = 1.5314 \\
& n_3^e = 1.4804 \\
& \theta_m = 58.34^\circ
\end{aligned}
\]

\[
\begin{aligned}
& n_1^e = 1.4603 \\
& n_2^0 = 1.5314 \\
& n_3^e = 1.4804
\end{aligned}
\]