NONLINEAR OPTICS (PHYC/ECE 568)

Spring 2022 - Instructor: M. Sheik-Bahae

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Homework #3, Due: Monday March 7

Problem 1. *Infrared Up-Conversion*

Estimate the efficiency of the upconversion of 10 μm infrared radiation using sum frequency generation. The pump laser has λ=532 nm, with a power of 10 W. Use 2 cm long Proustite (use $d_{eff}=d_{22}$) crystal under perfect phase-matching and optimum focusing where $L=2Z_0$ (Z_0 is the Rayleigh range of the focused laser beam).

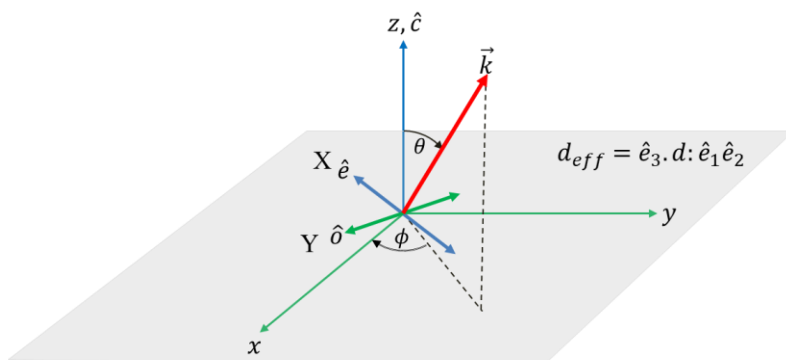
Problem 2.

(a) Show that d_{eff}, as defined by $P_3 = 4d_{eff} E_1 E_2$, is related to the d tensor via:

$$d_{eff} = \hat{e}_3 \bullet d{:}\hat{e}_1\hat{e}_2$$

where $\hat{e}_j$ *(j=1,2,3)* is the unit vector associated with E_1, E_2, and P_3.

(b) For a given geometry, d_{eff} is usually calculated in terms of *d_{il}'s* and the angles ϕ and θ as described in the figure below. Here *x, y and z* (or *1,2 and 3*) are the crystal axis and X, Y, and Z (laboratory frame) are optical propagation axis (e.g. $k_2 \| k_1 \| Z$). Note: Y is in xy-plane (thus normal to z- or optics axis) and X is on zZ plane.



(i) Derive expressions for d_{eff} for a class 3m crystal (e.g. $LiNbO_3$) where $\hat{e}_1 = \hat{e}_2 = Y$ (ordinary), and $\hat{e}_3 = X$ (extra-ordinary). (This is known as type-I condition).

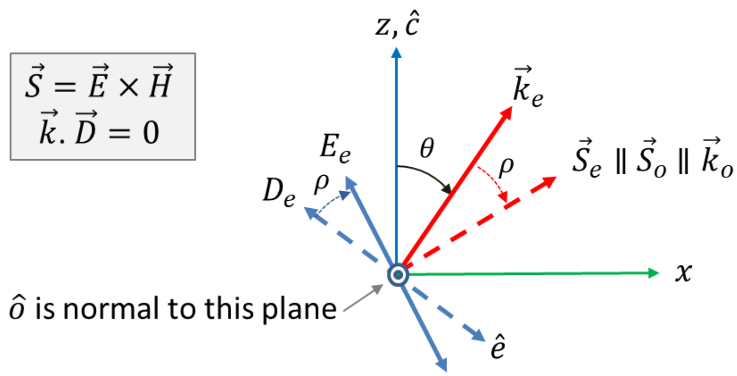
(ii) Repeat the above calculation for type-II condition where $\hat{e}_1 = Y$ (ordinary) , $\hat{e}_2 = X$ and $\hat{e}_3 = X$ (extra-ordinary).

(iii) Find a geometry (i.e. θ and ϕ) that accesses the largest d_{il} element in $LiNbO_3$. (see data provided here).

(iv) Find the phase matching angle for SHG generation at λ (fundamental)=1.15 μm for the part (i) and (ii). Discuss the phase matching situation for case (iii)

Problem 3. *Poynting Vector Walk-off:*

We know, from linear optics, that the e- and o-rays in a birefringent crystal walk-off from each other (i.e. double-refraction) resulting from the fact that k_e and k_o are not parallel. Known as Poynting vector walk-off, this is essentially the angle (ρ) between E and D vectors for the e-ray where D=ϵ: E.



In the harmonic generation applications, such as SHG, this imposes a serious restriction on the useful length of the nonlinear crystal.

a. Assuming a uniaxial crystal, calculate the walk-off angle ρ between e- and o-ray Poynting vectors. Show that

$$\tan\rho = -\frac{1}{n_e(\theta)}\frac{dn_e(\theta)}{d\theta}$$

b. Show that for type-I phase matching SHG (o+o→e)

$$\rho \approx \tan\rho = \frac{n_o^2(\omega)}{2}\left[\frac{1}{\tilde{n}_e^2(2\omega)} - \frac{1}{n_o^2(2\omega)}\right]\sin(2\theta_m)$$

$LiNbO_3$ Properties

$$\begin{bmatrix} 0 & 0 & 0 & 0 & d_{31} & -d_{22} \\ -d_{22} & d_{22} & 0 & d_{31} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}$$ ***class 3m crystal***

Nonlinear Optical Coefficients of $LiNbO_3$ at 1.06 μ m

d_{22} / l d_{36}^{KDP} l	6.5
d_{31} / l d_{36}^{KDP} l	-12.3
d_{33} / l d_{36}^{KDP} l	-86

d_{36} (KDP). = 0.4 pm/V

Refractive Indices at 20°C

Wavelength, µm	n_o	n_e
0.43584	2.39276	2.29278
0.54608	2.31657	2.22816
0.63282	2.28647	2.20240
1.1523	2.2273	2.1515
3.3913	2.1451	2.0822

NLO - PS # 2 Solution

Problem (1) $\rightarrow$

Conversion (quantum) efficiency defines as:

$$\begin{cases} \eta_1 = 1 - \left| \dfrac{A_1(L)}{A_1(0)} \right|^2 & \text{or} \\ \eta_2 = \left| \dfrac{A_3(L)}{A_1(0)} \right|^2 = \eta_1 \dfrac{\omega_3}{\omega_1} \end{cases}$$

Boyd (2.4.7) $\rightarrow$ $\eta_1 = 1 - \cos^2(KL) = \sin^2(KL)$

$$K = \frac{8\pi d}{c} \sqrt{\frac{\omega_1 \omega_3}{n_1 n_3}} |A_2| \rightarrow$$

- in order to have K in SI, we should set $\begin{cases} 4\pi\epsilon_0 \rightarrow 1 \\ d \rightarrow \epsilon_0 d \end{cases}$

$$\rightarrow K = \frac{2d}{c} \sqrt{\frac{\omega_1 \omega_3}{n_1 n_3}} |A_2| \Rightarrow K^2 = \frac{2 d^2 \omega_1 \omega_3}{c^3 \epsilon_0 n_1 n_2 n_3} I_2$$

$\rightarrow$ Also, we know: $I_2 = 2 n_2 \epsilon_0 c |A_2|^2$

$$I_2 = \frac{P_2}{\text{Area}} = \frac{2P_2}{\pi \omega_0^2},$$

for optimum focusing: $L = 2z_0 = \dfrac{2\pi n_2 \omega_0^2}{\lambda_2} = \dfrac{n_2 \omega_2 \omega_0^2}{c}$

- from above relations: $I_2 = \dfrac{2P_2 n_2 \omega_2}{\pi c L}$

$$\Rightarrow K^2 L^2 = \frac{4 d^2 \omega_1 \omega_2 \omega_3}{\pi c^4 \epsilon_0 n_1 n_3} P_2 L = \frac{32 d^2 P_2 L}{\pi \epsilon_0 c \lambda_1 \lambda_2 \lambda_3 n_1 n_3} ; \quad \lambda_3 = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$

$$\Rightarrow K^2 L^2 = \frac{32 d^2 P_2 L (\lambda_1 + \lambda_2)}{\pi \epsilon_0 c (\lambda_1 \lambda_2)^2 n_1 n_3} \checkmark$$

$\rightarrow$ based on given value: $\eta_1 \simeq K^2 L^2 \simeq 3.197\ \%$ $\checkmark$ (for $d = 16$ pm/V) problem (2) $\rightarrow$

(a)

$\vec{P}_3 = 4d : \vec{E}_1 \vec{E}_2$

$P_3 \hat{e}_3 = 4d : \hat{e}_1 \hat{e}_2 \, E_1 E_2$ $\rightsquigarrow$ Apply $(\hat{e}_3 \cdot)$ on both sides :

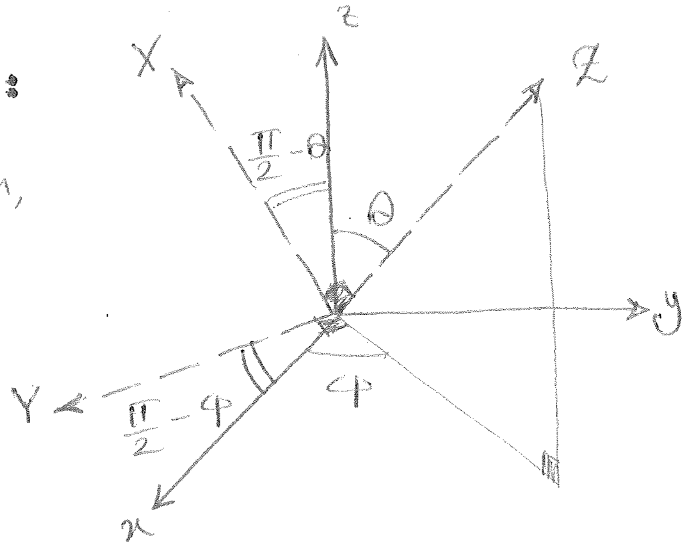
$P_3 (\hat{e}_3 \cdot \hat{e}_3) = 4\hat{e}_3 \cdot d : \hat{e}_1 \hat{e}_2 \, E_1 E_2$

$\rightarrow P_3 = 4 d_{eff} E_1 E_2$; where $d_{eff} = \hat{e}_3 \cdot d : \hat{e}_1 \hat{e}_2$ ✓

(b-i)

$\hat{e}_1 = \hat{e}_2 = X$ & $\hat{e}_3 = Y$

$$\begin{cases} X = -\cos\theta \cdot \cos\varphi \, \hat{x} - \cos\theta \sin\varphi \, \hat{y} + \sin\theta \, \hat{z} \\ Y = \sin\varphi \, \hat{x} - \cos\varphi \, \hat{y} \end{cases}$$



* $d : \hat{e}_1 \hat{e}_2$ is evaluated as: below:

- after applying $\hat{e}_3 \cdot (\;)$ operation, there only $\hat{x}$ & $\hat{y}$ components of $d : \hat{e}_1 \hat{e}_2$ remain!

$$\begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{bmatrix} \begin{bmatrix} \cos^2\theta \cos^2\varphi \\ \cos^2\theta \sin^2\varphi \\ \sin^2\theta \\ -2\sin\theta\cos\theta\sin\varphi \\ -2\sin\theta\cos\theta\cos\varphi \\ 2\cos^2\theta \sin\varphi\cos\varphi \\ \\ \end{bmatrix}$$

$$\rightarrow d_{eff} = \cos^2\theta \cos^2\varphi \left(d_{11} \sin\varphi - d_{21} \cos\varphi \right) +$$
$$\cos^2\theta \sin^2\varphi \left(d_{12} \sin\varphi - d_{22} \cos\varphi \right) +$$
$$\sin^2\theta \left(d_{13} \sin\varphi - d_{23} \cos\varphi \right) +$$
$$- \sin 2\theta \sin\varphi \left(d_{14} \sin\varphi - d_{24} \cos\varphi \right) +$$
$$- \sin 2\theta \cos\varphi \left(d_{15} \sin\varphi - d_{25} \cos\varphi \right) +$$
$$\cos^2\theta \sin 2\varphi \left(d_{16} \sin\varphi - d_{26} \cos\varphi \right).$$

* for a $\underline{3m}$ crystal :

$$d = \begin{vmatrix} 0 & 0 & 0 & 0 & d_{31} & -d_{22} \\ -d_{22} & d_{22} & 0 & d_{31} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{vmatrix}$$

$$\rightarrow d_{eff} = \left[\cos^2\theta \cos^2\varphi \cos\varphi - \cos^2\theta \sin^2\varphi \cos\varphi - \cos^2\theta \sin 2\varphi \sin\varphi \right] d_{22} +$$
$$+ \left[\sin 2\theta \sin\varphi \cos\varphi - \sin 2\theta \cos\varphi \sin\varphi \right] d_{31}$$
$$= \cos^2\theta \left[\cos\varphi \left(\cos 2\varphi \right) - \sin 2\varphi \sin\varphi \right] d_{22}$$
$$= \underline{\cos^2\theta \cos 3\varphi \, d_{22}}$$

- for the case of negative birefringent crystal,

$\hat{e}_1 = \hat{e}_2 = \hat{Y}$

$\hat{e}_3 = \hat{X}$

(HW question)

$d : \hat{e}_1 \hat{e}_2 =$

$$= \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{16} \\ d_{21} & d_{22} & \cdots & d_{26} \\ d_{31} & d_{32} & \cdots & d_{36} \end{pmatrix} \begin{pmatrix} \sin^2\varphi \\ \cos^2\varphi \\ 0 \\ 0 \\ 0 \\ -2\sin\theta\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} d_{11}\sin^2\varphi + d_{22}\cos^2\varphi - d_{16}\sin 2\theta \\ d_{21}\sin^2\varphi + d_{22}\cos^2\varphi - d_{26}\sin 2\varphi \\ d_{31}\sin^2\varphi + d_{32}\cos^2\varphi - d_{36}\sin 2\varphi \end{pmatrix} \rightsquigarrow \text{apply } (\hat{e}_3 \cdot \ ) \rightarrow$$

$$d_{eff} = \left(d_{11}\sin^2\varphi + d_{12}\cos^2\varphi - d_{16}\sin 2\varphi\right)\left(-\cos\theta\cos\varphi\right) +$$
$$\left(d_{21}\sin^2\varphi + d_{22}\cos^2\varphi - d_{26}\sin 2\varphi\right)\left(-\cos\theta\sin\varphi\right) +$$
$$\left(d_{31}\sin^2\varphi + d_{32}\cos^2\varphi - d_{36}\sin 2\varphi\right)\left(\sin\theta\right).$$

* for $\underline{3m}$ crystal :

$$d_{eff} = -d_{22}\left[\sin 2\varphi\cos\theta\cos\varphi + \left(\sin^2\varphi - \cos^2\varphi\right)\left(-\cos\theta\sin\varphi\right) + \right.$$
$$\left. + d_{31}\left[\sin^2\varphi + \cos^2\varphi\right]\sin\theta \right. .$$
$$= -d_{22}\left[\cos\theta\left(\cos 2\theta\sin\varphi + \sin 2\varphi\cos\varphi\right)\right] + d_{31}\sin\theta$$

$$\rightarrow \boxed{d_{eff} = -d_{22}\cos\theta\sin 3\varphi + d_{31}\sin\theta} \quad \checkmark$$

(b-ii)

we can consider 2 cases: $\begin{cases} \text{Case-1} : \hat{e}_1 = \hat{X},\ \hat{e}_2 = \hat{Y},\ \hat{e}_3 = \hat{X} \\ \text{Case-2} : e_1 = \hat{X},\ \hat{e}_2 = \hat{Y},\ \hat{e}_3 = \hat{Y} \end{cases}$

$d : \hat{e}_1 \hat{e}_2 =$

$$\begin{bmatrix} d_{11} & \dots & d_{16} \\ d_{21} & \dots & d_{26} \\ d_{31} & \dots & d_{36} \end{bmatrix} \begin{bmatrix} -\cos\theta \cos\varphi \sin\varphi \\ \cos\theta \sin\varphi \cos\varphi \\ 0 \\ 0 \\ 0 \\ \cos\varphi\cos\theta - \sin\varphi\cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cos\theta \sin 2\varphi\,(d_{12}-d_{11}) + \cos\theta\cos 2\varphi\, d_{16} \\ \frac{1}{2}\cos\theta \sin 2\varphi\,(d_{22}-d_{21}) + \cos\theta\cos 2\varphi\, d_{26} \\ \frac{1}{2}\cos\theta \sin 2\varphi\,(d_{32}-d_{31}) + \cos\theta\cos 2\varphi\, d_{36} \end{bmatrix}$$

* applying $(\hat{e}_3 \cdot)$:

$\mapsto \hat{e}_3 = \hat{X}$: Case 1

$$d_{eff} = \frac{1}{2}\cos\theta \sin 2\varphi \left[-\cos\theta\cos\varphi\,(d_{12}-d_{11}) - \cos\theta \sin\varphi\,(d_{22}-d_{21}) + \sin\theta\,(d_{32}-d_{31}) \right]$$
$$+ \cos\theta\cos 2\varphi \left[-\cos\theta\cos\varphi\, d_{16} - \cos\theta\sin\varphi\, d_{26} + \sin\theta\, d_{36} \right]$$

$\mapsto \hat{e}_3 = \hat{Y}$: Case 2

$$d_{eff} = \frac{1}{2}\cos\theta \sin 2\varphi \left[\sin\varphi\,(d_{12}-d_{11}) - \cos\varphi\,(d_{22}-d_{21}) \right]$$
$$+ \cos\theta\cos 2\varphi \left[\sin\varphi\, d_{16} - \cos\varphi\, d_{26} \right]$$

* for a 3m crystal :

$$\text{Case-1} : d_{eff} = \frac{1}{2}\cos\theta \sin 2\varphi \cos\theta \sin\varphi\,(-2d_{22}) +$$
$$- \cos\theta\cos 2\varphi \cos\theta\cos\varphi\,(-d_{22}).$$
$$= d_{22}\cos^2\theta \cos 3\varphi$$

$\rightarrow$ H-W question

Case - 2 : $d_{eff} = -\frac{1}{2} \cos\theta \sin 2\varphi \cos\varphi \, (2d_{22}) +$

$\cos\theta \cos 2\varphi \sin\varphi \, (-d_{22})$.

$$= -d_{22} \cos\theta \sin 3\varphi$$

(b) - (iii)

- to access d_{33} (largest in $LiNbO_3$), one needs :

$$\hat{e}_1 \parallel \hat{e}_2 \parallel \underbrace{\hat{e}_3 \parallel X \parallel z}$$

$$\Downarrow$$

$\Rightarrow$ if $\theta = \frac{\pi}{2} \rightarrow d_{eff} = d_{33}$

- we shall see that this configuration is not phase-matched using the crystal birefringent. But, employing a growth technique know as "Periodic Polling" we can enhance phase matching under above geometry!

(b) - (iv)

$\lambda_1 = 1.15 \ \mu m$, $\lambda_2 = \frac{\lambda_1}{2} = 0.570$

we know for,

$\lambda = 0.632 \rightarrow n_o = 2.2864$, $n_e = 2.2024$

$\lambda = 0.546 \rightarrow n_o = 2.3165$, $n_e = 2.2281$

- Using linear interpolation of indices at these two wavelengths, we'll obtain n_o & n_e for $\lambda_2 = 0.570$:

$n_o(\lambda_2) = 2.306$

$n_e(\lambda_2) = 2.219$

Type - I PM :

$$n_e(\theta_m, \lambda_2) = n_o(\lambda_1) \rightarrow$$

$$\sin^2\theta_m = \frac{n_o^{-2}(\lambda_1) - n_o^{-2}(\lambda_2)}{n_e^{-2}(\lambda_2) - n_o^{-2}(\lambda_2)} = 0.906 \rightarrow \theta_m = 72^\circ$$

Type - II PM :

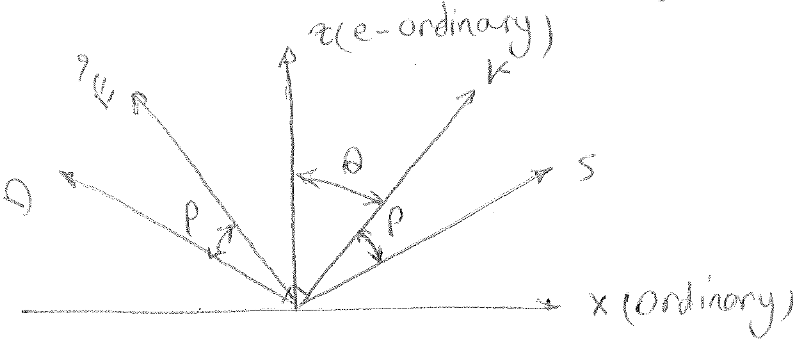
$$n_o(\lambda_1) + n_e(\theta_m, \lambda_1) = \frac{\lambda_1}{\lambda_2} n_e(\theta_m, \lambda_2)$$

- Upon inserting the numbers, one finds (using either graphical or math softwares) that there is no solution to this. That is, this condition is not type-II phase matchable.

Accessing d_{33} required that $\theta = 90^\circ$. This condition is not phase matchable for this wavelength. Quasi Phase Matching (QPM) is ideal as has been used in periodically polled lithium niobate (PPLN).

problem (3)

- The poynting vector walk-off for a uniaxial crystal can be treated in a two-dimentional analysis.



- Consider the extra-ordinary ray. (i.e. $\theta \neq 0$), for this case, $\vec{K} \perp \vec{D}$ & $\vec{S} \perp \vec{E}$, the walk-off angle ρ is also the angle between $\vec{E}$ & $\vec{D}$!

$$\rightarrow \vec{D}.\vec{E} = |\vec{D}| \cdot |\vec{E}| \cos\rho \longrightarrow \cos\rho = \frac{\vec{D}.\vec{E}}{|\vec{D}| \cdot |\vec{E}|}$$

$$\rightarrow \cos\rho = \frac{D_x E_x + D_z E_z}{(D_x^2 + D_z^2)^{1/2} (E_x^2 + E_z^2)^{1/2}}$$

but $D_x = -D \cos\theta$

$D_z = D \sin\theta$

also $D_x = \epsilon_x E_x = n_o^2 E_x$

$D_z = \epsilon_z E_z = n_e^2 E_z$

$$\Rightarrow E_x = -\frac{D.\cos\theta}{n_o^2} \quad \& \quad E_z = \frac{D.\sin\theta}{n_e^2}$$

$$\rightarrow \cos\rho = \frac{D^2 \left[\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2} \right]}{D^2 \left[\frac{\cos^2\theta}{n_o^4} + \frac{\sin^2\theta}{n_e^4} \right]^{1/2}} \rightarrow \cos^2\rho = \frac{\left(\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2} \right)^2}{\frac{\cos^2\theta}{n_o^4} + \frac{\sin^2\theta}{n_e^4}}$$

$$\rightarrow \tan^2\rho = \frac{1}{\cos^2\rho} - 1$$

$$= \left(\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}\right)^{-2}\left[\frac{\cos^2\theta}{n_o^4} + \frac{\sin^2\theta}{n_e^4} - \frac{\cos^4\theta}{n_o^4} - \frac{\sin^4\theta}{n_e^4} - \frac{2\sin^2\theta\cos^2\theta}{n_o^2 n_e^2}\right]$$

$$\rightarrow \tan\rho = \frac{\cos\theta\sin\theta\left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)}{\left(\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}\right)}$$

$$= \frac{1}{2}\left[\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}\right]^{-1}\left(\frac{1}{n_o^2} - \frac{1}{n_e^2}\right)\sin 2\theta \quad \checkmark$$

(b)

for Type-I SHG, where $o + o \rightarrow e$

$$\frac{1}{n_o^2(\omega)} = \frac{\cos^2\theta_m}{n_o^2(2\omega)} + \frac{\sin^2\theta_m}{n_e^2(2\omega)}$$

thus,

$$\tan\rho = \frac{1}{2} n_o^2(\omega) \times \left[\frac{1}{n_o^2(2\omega)} - \frac{1}{n_e^2(2\omega)}\right]\sin 2\theta_m \quad \checkmark$$

Solution for second part of 3(a) in HW22_3 :

$$\frac{1}{n_e(\theta)^2} = \frac{\sin(\theta)^2}{\tilde{n}_e^2} + \frac{\cos(\theta)^2}{n_o^2}$$

Take derivative with respect to θ:

$$\frac{-2\frac{dn_e(\theta)}{d\theta}}{n_e(\theta)^3} = 2\sin(\theta)\cos(\theta)\left(\frac{1}{\tilde{n}_e^2} - \frac{1}{n_o^2}\right) = \sin(2\theta)\left(\frac{1}{\tilde{n}_e^2} - \frac{1}{n_o^2}\right)$$

Therefore

$$-\frac{1}{n_e(\theta)}\frac{dn_e(\theta)}{d\theta} = \frac{1}{2}n_e(\theta)^2\sin(2\theta) = \tan(\rho)$$