1. Nearly Degenerate FWM
Consider the FWM (forward) geometry shown in the figure below where two pump beams at frequency \(\omega_1\) and \(\omega_2\) (\(\omega_1=\omega_2+\delta\), \(\delta\) small) are incident on a \(\chi^{(3)}\) material to produce a signal at a frequency \(\omega_3=2\omega_1-\omega_2\). For very small angles \(\theta\), what is direction (i.e. the angle) with which the generated signal at \(\omega_3\) will be propagating? Note: \(\delta\) is small enough to ignore dispersion.

2. Z-Scan (See also Problem 7.3 in Boyd)
A thin nonlinear optical material having a thickness \(L\) and a nonlinear index coefficient \(n_2\) is scanned along \(z\) (propagation direction) near the focus of a Gaussian beam that is characterized by its minimum spot size \(w_0\), wavelength \(\lambda_0\) and power \(P\) (see Figure). We know, from aberration-free approximation, that the induced Kerr-lens focal length \(f_{nl}(z) = aw^2(z)/4Ln_2I(z)\) where \(I(z)=2P/\pi w^2(z)\) is the on-axis intensity, \(w(z)=w_0(1+z^2/z_0^2)^{1/2}\) is the beam radius and \(a\) is a correction factor (constant).

(a) Use the ABCD matrix formalism to derive expressions for the beam radius \(w_a\) and on-axis intensity \(I_a\) (=2\(P/\pi w_a^2\)) at a distance \(d\) from the focus.

(b) Simplify the expression for \(I_a\) (obtained in (a)) by retaining the lowest order nonlinear term in the power expansion and by assuming that \(d>>Z,Z_0\). You should be able to express your results in terms of \(x=z/z_0\) and \(\Delta\Phi_0/a\) where \(\Delta\Phi_0=(2\pi/\lambda_0)n_2I(0)L\) is the on-axis nonlinear phase shift at the focus.

(c) By placing a small on-axis aperture and a detector at the observation plane we can measure \(I_a\). Obtain an expression for the normalized transmittance \(T(x, \Delta\Phi_0) = I_a(\Delta\Phi_0, x)/I_a(0, x)\). Plot \(T\) versus \(x\) for \(-4<x<4\).

(d) Compare your results in (c) with that obtained using the diffraction theory:

\[
T(\Delta\Phi_0, x) \approx 1 + \frac{4x\Delta\Phi_0}{(1 + x^2)(9 + x^2)}
\]

What value of \(a\) gives a good overall agreement between the two approaches.