1. Nearly Degenerate FWM
Consider the FWM (forward) geometry shown in the figure below where two pump beams at frequency $\omega_1$ and $\omega_2$ ($\omega_1=\omega_2+\delta$, $\delta$ small) are incident on a $\chi^{(3)}$ material to produce a signal at a frequency $\omega_3=2\omega_1-\omega_2$. For very small angles $\theta$, what is direction (i.e., the angle) with which the generated signal at $\omega_3$ will be propagating? Note: $\delta$ is small enough to ignore dispersion.

2. Self-Phase Modulation (SPM)
   a. A laser pulse with an intensity profile $I=I_0 \text{Sech}^2(t/\tau_0)$ having $\tau_0=200$ fs and $\lambda=500$ nm, and $I_0=1$ GW/cm$^2$ is coupled into a silica fiber having an instantaneous (ultrafast) $n_2=2 \times 10^{-16}$ cm$^2$/W. Estimate the required length of fiber ($L$) for the spectrum of the pulse to broaden (due to SPM only) to $\approx 5$ times its original value.

   b. Using linear dispersive elements such as grating pairs, we can compress the exit pulse in part (a) to a transform-limited pulse having a width $\approx 1/\Delta \omega$. Estimate the resultant compressed pulse width.

   c. If the pulse in (a) propagates in a nonlinear medium with $n_2=1 \times 10^{-14}$ cm$^2$/W but with a relaxation time $\tau=2$ ps (i.e., nonlinear refraction accumulates), what is the required length to double its spectrum. Qualitatively, describe the transmitted spectrum as compared to that of part (a).

3. Pick one of the following two problems (next 2 pages) on Intensity Dependent Refractive Index:
3(a) Beam Deflection

As briefly discussed in the class, among the sensitive methods of nonlinear refraction measurements is the beam deflection technique. As shown in the figure, a pump beam (with a Gaussian spatial profile) induces a refractive index profile in the nonlinear material. A weak probe beam—with a diameter much smaller than that of the pump—centered at a distance $x_0$ away from the pump beam will be therefore deflected by this index gradient.

Assume a pump irradiance $I(r)=I_0 \exp(-2r^2/w_0^2)$ and optical Kerr effect: $n=n_0+n_2I$. We also know that a ray traveling through a thin material having a transverse gradient index $n(r)$ is deflected by an angle $\phi=\nabla n(r)L/n_0$ where $L$ is the thickness of the sample and $n_0$ is the linear refractive index.

a) What is the optimum position $x_0$ for which the maximum deflection occurs?

b) What is the maximum deflection angle?

The deflection of the beam is measured using a “bi-cell” detector in the far field as shown in the figure. In that case, the measured normalized signal is $S=(V_1-V_2)/(V_1+V_2)$.

c) Show that

$$S = \frac{\sqrt{8\pi}}{\lambda} \frac{W_0}{\phi}$$

where $W_0$ is the probe beam radius at the sample and $\lambda$ is the wavelength.

d) Write $S$ in terms of the maximum on axis phase shift $\Delta \Phi_0=(2\pi/\lambda)n_2I_0L$ (i.e. $S=K\Delta \Phi_0$). Derive an expression for $K$ and compare the sensitivity of this technique with that of z-scan where the normalized peak-to-valley transmittance signal is given by $\Delta T_{pv}\approx 0.4\Delta \Phi$ for $w_0 \approx w_0$, $n_0 = 2$. 

![Diagram of beam deflection](image)
3(b) Z-Scan (See also Problem 7.3 in Boyd)

A thin nonlinear optical material having a thickness \( L \) and a nonlinear index coefficient \( n_2 \) is scanned along \( z \) (propagation direction) near the focus of a Gaussian beam that is characterized by its minimum spot size \( w_0 \), wavelength \( \lambda_0 \) and power \( P \) (see Figure). We know, from aberration-free approximation, that the induced Kerr-lens focal length \( f_{nl}(z) = aw^2(z)/4Ln_2I(z) \) where \( I(z) = 2P/\pi w_0^2(z) \) is the on-axis intensity, \( w(z)^2 = w_0^2(1 + z^2/z_0^2) \), and \( a \) is a numerical correction factor.

(a) Use the ABCD matrix formalism to derive expressions for the beam radius \( w_a \) and on-axis intensity \( I_a = 2P/\pi w_a^2 \) at a distance \( d \) from the focus.

(b) Simplify the expression for \( I_a \) (obtained in (a)) by retaining the lowest order nonlinear term in the power expansion and by assuming that \( d >> Z, Z_0 \). You should be able to express your results in terms of \( x = z/z_0 \) and \( \Delta \Phi_0/a \) where \( \Delta \Phi_0 = (2\pi/\lambda_0)n_2I(0)L \) is the on-axis nonlinear phase shift at the focus.

(c) By placing a small on-axis aperture and a detector at the observation plane we can measure \( I_a \). Obtain an expression for the normalized transmittance \( T(x, \Delta \Phi_0) = I_a(\Delta \Phi_0, x)/I_a(0, x) \). Plot \( T \) versus \( x \) for \(-4 < x < 4\).

(d) Compare your results in (c) with that obtained using the diffraction theory:

\[
T(\Delta \Phi_0, x) \approx 1 + \frac{4x\Delta \Phi_0}{(1 + x^2)(9 + x^2)}
\]

What value of \( a \) (the correction factor) gives a good overall agreement between the two approaches.