Laser Physics I (PHYC/ECE 464)

Homework #4, Due Monday, Sept. 26, Fall 2022

1 Consider a linear combination of two equal amplitude $\mathrm{TEM}_{m,p}$ modes given by:

$$\mathbf{E} = E_0 \left\{ (\mathrm{TEM}_{1,0})\mathbf{a}_y \pm j(\mathrm{TEM}_{0,1})\mathbf{a}_x \right\}$$

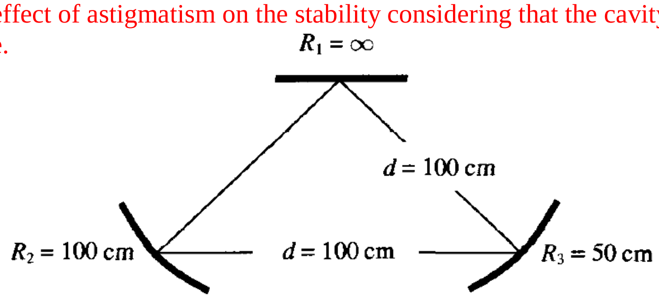
(a) Sketch the "dot" pattern or equal intensity contours for each component (i.e., $\mathbf{a}_x$ or $\mathbf{a}_y$). Indicate the direction of the electric field.

(b) Sketch the pattern for the linear combination.

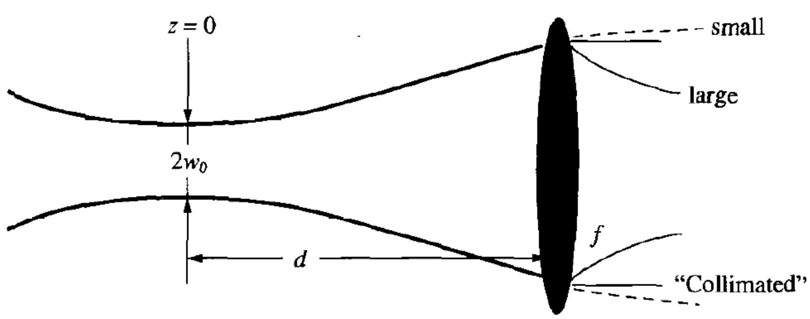
(c) Label the positions where the intensity is a maximum and a minimum. (This is sometimes referred to as the "donut mode" or $\mathrm{TEM}^*_{0,1}$.

2 Is the cavity shown below stable? Demonstrate the logic of your answer by (a) constructing a unit cell starting at the flat mirror, (b) finding the $ABCD$ matrix for that cell, and (c) applying the stability criteria. (d) What are the circumstances under which the quantity $[AD - BC]$ can be different from 1? Why is $AD - BC$ always equal to 1 for a cavity? (Ignore astigmatism)

Bonus: Show the effect of astigmatism on the stability considering that the cavity axis below forms an equilateral triangle.



3 A focused Gaussian beam reaches its minimum spot size w_0 at $z = 0$ where $R = \infty$ and then propagates to a thin lens of focal length f located a distance d from $z = 0$. If w_0 is large, then the beam exiting the lens will be focused. If it is too small, then the lens merely reduces the far field spreading angle. Find the critical value of w_0 such that the output beam is "collimated"; that is, $R(z = d^+) = \infty$ also.



4. Using Eq. (5.2.8), write z_0^2 in terms of the g-paramters (i.e. g_1 and g_2). From this, derive the stability condition of the cavity in terms of g_1 and g_2. Compare this with the geometric optics results obtained eralier.

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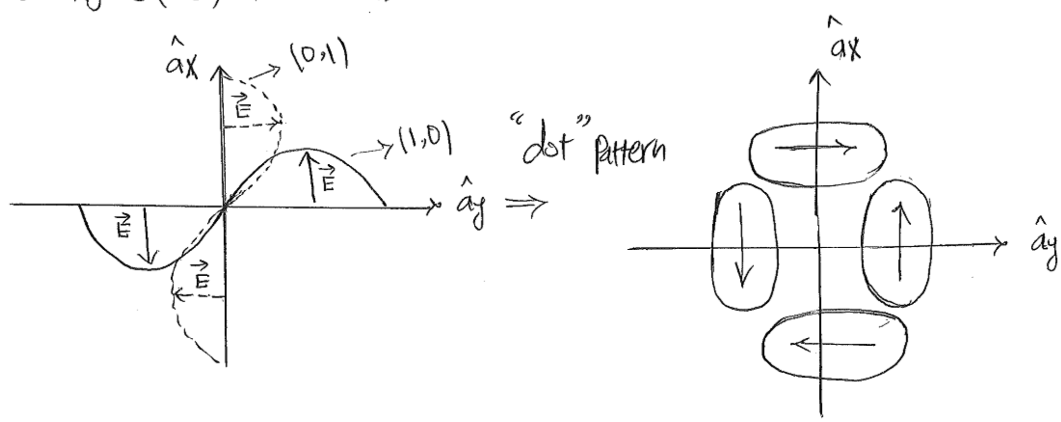
Problem (3.6) Verdeyen 3^{rd} ed.

$$\vec{E} = E_0\left((TEM_{1,0})\,\hat{a}_y \pm j\,(TEM_{0,1})\,\hat{a}_x\right)$$

(a) $$E(x,y,z) = E_{m,p}\, H_m\!\left(\frac{\sqrt{2}\,x}{w(z)}\right) H_p\!\left(\frac{\sqrt{2}\,y}{w(z)}\right) \frac{w_0}{w(z)}\, e^{\frac{-(x^2+y^2)}{w^2(z)}} \times e^{-j(\text{phase})}$$

↳ Equation 3.5.1 of the text.

and we know that $H_0(u) = 1$ & $H_1(u) = 2u$ and according to the figure (3.5) of the text,

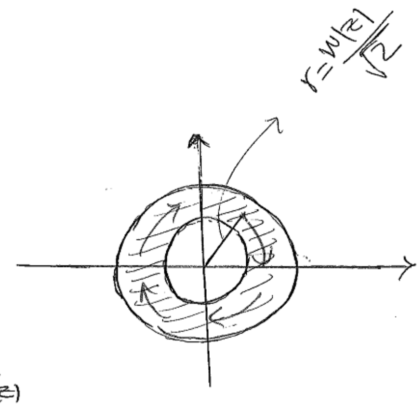


(b) $$|E|^2 = |E_0|^2\left((TEM_{10})^2 + (TEM_{01})^2\right)$$

$$\Rightarrow |E|^2 = |E_0|^2\left(2^2\left(\frac{\sqrt{2}\,x}{w(z)}\right)^2 \times 1 \times \left(\frac{w_0}{w(z)}\right)^2 e^{\frac{-2r^2}{w(z)^2}} + 1 \times 2^2\left(\frac{\sqrt{2}\,y}{w(z)}\right)^2\left(\frac{w_0}{w(z)}\right)^2 e^{\frac{-2r^2}{w^2(z)}}\right)$$

$$= \overbrace{|E_0|^2\, 4\left(\frac{\sqrt{2}}{w(z)}\right)^2\left(\frac{w_0}{w(z)}\right)^2}^{\alpha}\left(x^2+y^2\right) e^{\frac{-2r^2}{w^2(z)}}$$

$$= \alpha\, r^2 e^{-2r^2/w^2(z)}$$



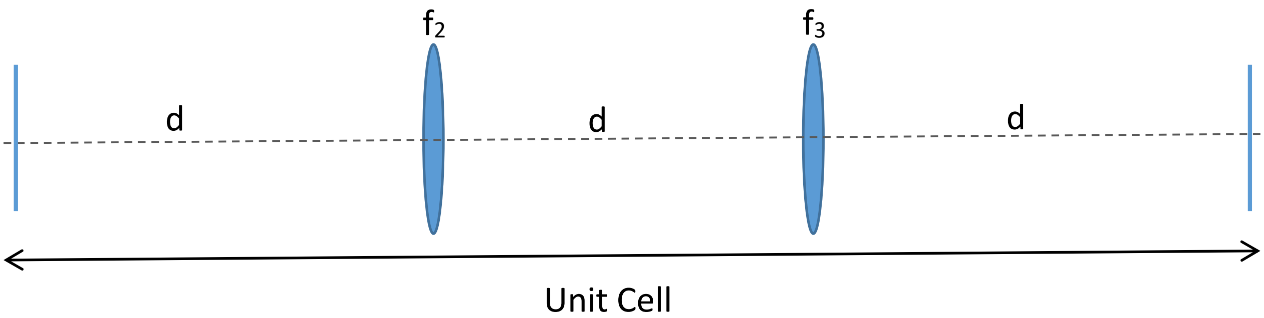
$$I \sim |E|^2 \sim r^2 e^{-2r^2/w(z)^2}$$

$$\frac{dI}{dr} \sim \frac{d}{dr}\left(\alpha r^2 e^{-2r^2/w(z)^2}\right) = \alpha\left(2r\, e^{\frac{-2r^2}{w(z)^2}} - \frac{4r}{w(z)^2} r^2 e^{-2r^2/w(z)^2}\right) = 0$$

$$\Rightarrow 1 - \frac{2r^2}{w(z)^2} = 0 \Rightarrow \quad r = \frac{w(z)}{\sqrt{2}}$$ (Maximum occurs at this radius).

$$\left(\left.\frac{d^2 I}{dr^2}\right|_{r=\frac{w(z)}{2}} < 0\right)$$

Verdeyen 3.18: Ignoring the astigmatism introduced by the oblique incidence on curved mirrors



The equivalent transmission system for the ring cavity is

$$T = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ -1/f_3 & 1 - d/f_3 \end{pmatrix} \begin{pmatrix} 1 & d \\ -1/f_2 & 1 - d/f_2 \end{pmatrix} = \begin{pmatrix} 1 - \frac{2d}{f_2} - \frac{d}{f_3} + \frac{d^2}{f_2 f_3} & something \\ something & 1 - \frac{2d}{f_3} - \frac{d}{f_2} + \frac{d^2}{f_2 f_3} \end{pmatrix}$$

If d=100 cm, f_2=50cm, f_3=25 cm, then A=+1 and D=-1, and $S = \frac{A+D+2}{4} = \frac{1}{2}$ (>0 and <1); the cavity is therefore stable.

If the entrance and exit planes have different indices of refraction, the AD-BC=$n_2/n_1 \neq 1$.

In a cavity (roundtrip unit cell), the start and stop planes are always the same, thus AD-BC=1 always.

BONUS PART: With astigmatism, the angle of incidence on each mirror is θ=60/2 degrees. Then, f's above will need to change to f/cos(θ) and f*cos(θ) for sagittal and tangential components respectively.

This leads to S=0.103 (for sagittal, stable) and S=1.137 (tangential, unstable). The cavity then as a whole is not stable.

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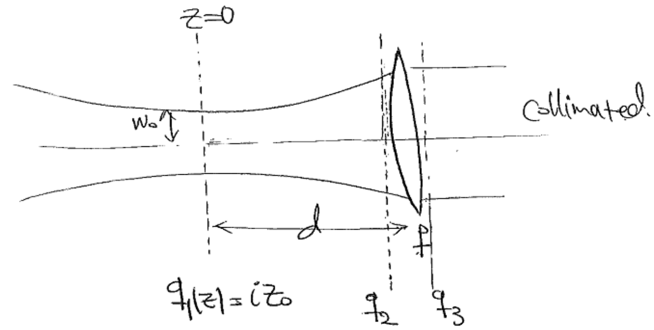
Problem (3.21) (Verdeyen 3rd ed.)



$$\left(z_0 = \frac{\pi w_0^2}{\lambda_0} \right)$$

The radius of the beam at $z=d$ is,

$$R(z) = z\left(1+\left(\frac{z_0}{z}\right)^2\right) \Rightarrow R_2(z=d) = d\left(1+\left(\frac{z_0}{d}\right)^2\right) = \frac{z_0^2+d^2}{d}$$

The transformation matrix for a thin lens $M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

$$\frac{1}{q_3} = \frac{C+D(1/q_2)}{A+B(1/q_2)} = \frac{-1/f + \frac{1}{q_2}}{1+0} = -\frac{1}{f} + \frac{1}{q_2}$$

$$= -\frac{1}{f} + \overbrace{\frac{1}{R_2} - j\frac{\lambda_0}{\pi W_2^2}}$$

on the other hand,

$$\frac{1}{q_3} = \frac{1}{R_3} - j\frac{\lambda_0}{\pi W_3^2} = \frac{-1}{f} + \frac{1}{R_2} - j\frac{\lambda_0}{\pi W_2^2}$$

$$\Rightarrow \frac{1}{R_3} = \frac{-1}{f} + \frac{1}{R_2} \Rightarrow R_3 = \frac{R_2 f}{R_2 - f} \xrightarrow{R_3 \to \infty} R_2 = f$$

$$R_2 = \frac{d^2+z_0^2}{d} = f \Rightarrow z_0^2 = fd - d^2$$

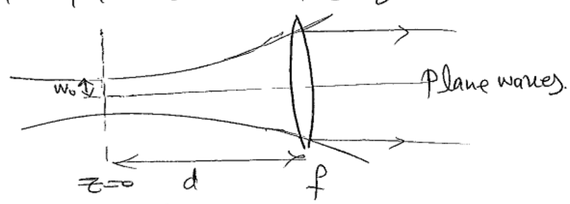
$$z_0 = \frac{\pi W_0^2}{\lambda_0} \quad ; \quad W_0^4 = \frac{\lambda_0^2}{\pi^2}\left(fd - d^2\right)$$

$$\Rightarrow \boxed{W_0 = \sqrt{\frac{\lambda_0}{\pi}}\,\left(d(f-d)\right)^{1/4}}$$

So the beam waist must be this amount to have a plane wave after the lens. ($R_3 = \infty$)

Alternative way :

calculate the matrix from the beam waist $2W_0$ to the lens and then use the ABCD law to find the q parameter after the lens.



$$q_1 = \cancel{z}^{\,0} + iz_0 = iz_0$$

$$T = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d \\ -\frac{1}{f} & 1-d/f \end{pmatrix}$$

Now, use $$\frac{1}{q_2} = \frac{C + D(1/q_1)}{A + B(1/q_1)}$$

$$\Rightarrow \frac{1}{q_2} = \frac{-1/f + (1-d/f)\frac{1}{iz_0}}{1 + d/iz_0} = \frac{-iz_0/f + (1-d/f)}{iz_0 + d} \times \frac{-iz_0+d}{iz_0+d}$$

$$= \frac{d(1-d/f) - z_0^2/f - iz_0 d/f - iz_0(1-d/f)}{d^2+z_0^2}$$

$$= \frac{d(1-d/f) - z_0^2/f \;\; - i\left(\cancel{z_0 d/f} + z_0 - \cancel{z_0 d/f}\right)}{d^2+z_0^2}$$

on the other hand, $$\frac{1}{q_2} = \frac{1}{R_2(z)} - i\frac{\lambda_0}{\pi W_2^2(z)}$$

Equating $\frac{1}{q_2}$'s, $\underbrace{\text{we have,}}_{\text{for the real part}}$

$$\Rightarrow \frac{1}{R_2(z)} = \frac{d(1-d/f) - z_0^2/f}{d^2+z_0^2} \Rightarrow R_2(z) = \frac{d^2+z_0^2}{d(1-d/f) - z_0^2/f}$$

denominator must go to zero to have $R_2 \to \infty$.

$$d(1-d/f) - \frac{z_0^2}{f} = 0 \Rightarrow z_0^2 = d(f-d) \quad \left(z_0 = \frac{\pi W_0^2}{\lambda_0}\right)$$

$$\Rightarrow W_0^4 = \frac{\lambda_0^2}{\pi^2} d(f-d) \Rightarrow W_0 = \left(\frac{\lambda_0^2}{\pi^2} d(f-d)\right)^{1/4}$$

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$$z_0^2 = \frac{d(R_1-d)(R_2-d)(R_1+R_2-d)}{(R_1+R_2-2d)^2} \quad ; \quad g_{1,2} = 1 - \frac{d}{R_{1,2}} = \frac{R_{1,2}-d}{R_{1,2}}$$

$$\to z_0^2 = \frac{d(R_1 g_1)(R_2 g_2)(R_1+R_2-d)}{(g_1 R_1 + g_2 R_2)^2} =$$

$$= \frac{d^2 R_1 R_2 g_1 g_2 (1-g_1 g_2) / (1-g_1)(1-g_2)}{(g_1 R_1 + g_2 R_2)^2}$$

$$= \frac{d^4 \left(\frac{1}{1-g_1}\right)\left(\frac{1}{1-g_2}\right) g_1 g_2 (1-g_1 g_2)}{(1-g_1)(1-g_2)\left(\frac{g_1 d}{1-g_1} + \frac{g_2 d}{1-g_2}\right)^2} = \frac{d^4 g_1 g_2 (1-g_1 g_2)}{(1-g_1)^2(1-g_2)^2 \frac{[d(g_1+g_2-2g_1 g_2)]^2}{(1-g_1)^2(1-g_2)^2}}$$

$$\Rightarrow z_0^2 = \frac{d^2 g_1 g_2 (1-g_1 g_2)}{(g_1+g_2-2g_1 g_2)^2}$$

Stability Cond. $z_0^2 \geq 0 \to g_1 g_2 (1-g_1 g_2) \geq 0$

$$\Rightarrow \begin{cases} g_1 g_2 \geq 0 \\ 1 - g_1 g_2 \geq 0 \end{cases} \Rightarrow 0 \leq g_1 g_2 \leq 1$$