1. Nearly Degenerate FWM
Consider the FWM (forward) geometry shown in the figure below where two pump beams at frequency $\omega_1$ and $\omega_2$ ($\omega_1=\omega_2+\delta$, $\delta$ small) are incident on a $\chi^{(3)}$ material to produce a signal at a frequency $\omega_3=2\omega_1-\omega_2$. For very small angles $\theta$, what is direction (i.e. the angle) with which the generated signal at $\omega_3$ will be propagating? Note: $\delta$ is small enough to ignore dispersion.

2. Self-Phase Modulation (SPM)

a. A laser pulse with an intensity profile $I=I_0 \text{Sech}^2 (t/\tau_0)$ having $\tau_0=200$ fs and $\lambda=500$ nm, and $I_0=1$ GW/cm$^2$ is coupled into a silica fiber having an instantaneous (ultrafast) $n_2=2 \times 10^{-16}$ cm$^2$/W. Estimate the required length of fiber ($L$) for the spectrum of the pulse to broaden (due to SPM only) to $\approx 5$ times its original value.

b. Using linear dispersive elements such as grating pairs, we can compress the exit pulse in part (a) to a transform-limited pulse having a width $\approx 1/\Delta \omega$. Estimate the resultant compressed pulse width.

c. If the pulse in (a) propagates in a nonlinear medium with $n_2=1 \times 10^{-14}$ cm$^2$/W but with a relaxation time $\tau=2$ ps, what is the required length to double its spectrum? Qualitatively, describe the transmitted spectrum as compared to that of part (a).
Do either problem 3 or 4 (or both if you prefer!):

3. Intensity Dependent Refractive Index: Beam Deflection

As briefly discussed in the class, among the sensitive methods of nonlinear refraction measurements is the beam deflection technique. As shown in the figure, a pump beam (with a Gaussian spatial profile) induces a refractive index profile in the nonlinear material. A weak probe beam—with a diameter much smaller than that of the pump—centered at a distance \( x_0 \) away from the pump beam will be therefore deflected by this index gradient.

Assume a pump irradiance \( I(r)=I_0 \exp(-2r^2/w_e^2) \) and optical Kerr effect: \( n=n_0+n_2I \). We also know that a ray traveling through a thin material of transverse gradient index \( n(r) \) is deflected by an angle \( \phi=\nabla n(r)L/n_0 \) where \( L \) is the thickness of the sample and \( n_0 \) is the linear refractive index.

a) What is the optimum position \( x_0 \) for which the maximum deflection occurs?
b) What is the maximum deflection angle?

c) Show that
\[
S = \frac{\sqrt{8\pi} w_0}{\lambda} \phi
\]
where \( w_0 \) the probe beam radius at the sample and \( \lambda \) is the wavelength.
d) Write \( S \) in terms of the maximum on axis phase shift \( \Delta \Phi_0=(2\pi/\lambda)n_2I_0L \) (i.e. \( S=K\Delta \Phi_0 \)). Derive an expression for \( K \) and compare the sensitivity of this technique with that of z-scan where the normalized peak-to-valley transmittance signal is given by \( \Delta T_{pv}=0.4\Delta \Phi_0 \).
4. Z-Scan (See also Problem 7.3 in Boyd)

A thin nonlinear optical material having a thickness $L$ and a nonlinear index coefficient $n_2$ is scanned along $z$ (propagation direction) near the focus of a Gaussian beam that is characterized by its minimum spot size $w_0$, wavelength $\lambda_0$ and power $P$ (see Figure). We know, from aberration-free approximation, that the induced Kerr-lens focal length $f_{nl}(z) = aw^2(z)/4Ln_2I(z)$ where $I(z)=2P/\pi w^2(z)$ is the on-axis intensity, $w(z)=w_0(1+z^2/z_0^2)^{1/2}$ is the beam radius and $a$ is a correction factor (constant).

(a) Use the ABCD matrix formalism to derive expressions for the beam radius $w_a$ and on-axis intensity $I_a$ ($=2P/\pi w_a^2$) at a distance $d$ from the focus.

(b) Simplify the expression for $I_a$ (obtained in (a)) by retaining the lowest order nonlinear term in the power expansion and by assuming that $d>>z_0, z$. You should be able to express your results in terms of $x=z/z_0$ and $\Delta \Phi_0/a$ where $\Delta \Phi_0=(2\pi/\lambda_0)n_2I(0)L$ is the on-axis nonlinear phase shift at the focus.

(c) By placing a small on-axis aperture and a detector at the observation plane we can measure $I_a$. Obtain an expression for the normalized transmittance $T(x, \Delta \Phi_0) = I_a(x,\Delta \Phi_0)/I_a(0,x)$. Plot $T$ versus $x$ for $-4<x<4$.

(d) Compare your results in (c) with that obtained using the diffraction theory:

$$T(\Delta \Phi_0, x) \cong 1 + \frac{4x\Delta \Phi_0}{(1 + x^2)(9 + x^2)}$$

What value of $a$ gives a good overall agreement between the two approaches.